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Thermodynamic Treatment of Nonphysical Systems: Formalism and an Example (Single-Lane Traffic)

H. Reiss,[1] Audrey Dell Hammerich,[1] and E. W. Montroll[2]

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An effort is made to introduce thermodynamic and statistical thermodynamic methods into the treatment of nonphysical (e.g., social, economic, etc.) systems. Emphasis is placed on the use of the *entire* thermodynamic framework, not merely entropy. Entropy arises naturally, related in a simple manner to other measurables, but does not occupy a primary position in the theory. However, the maximum entropy formalism is a convenient procedure for deriving the thermodynamic analog framework in which undetermined multipliers are thermodynamic-like variables which summarize the collective behavior of the system. We discuss the analysis of Levine and his coworkers showing that the maximum entropy formalism is the unique algorithm for achieving consistent inference of probabilities. The thermodynamic-like formalism for treating a single lane of vehicular traffic is developed and applied to traffic in which the interaction between cars is chosen to be a particular form of the "follow-the-leader" type. The equation of state of the traffic, the distributions of velocity and headway, and the various thermodynamic-like parameters, e.g., temperature (collective sensitivity), pressure, etc. are determined for an experimental example (Holland Tunnel). Nearest-neighbor and pair correlation functions for the vehicles are also determined. Many interesting and suggestive results are obtained,

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1. ENTROPY AS A SECONDARY QUANTITY

For some time now investigators in a variety of fields have been searching for *entropy* in nonphysical systems. (See, for example, Georgescu-

[1] Department of Chemistry and Biochemistry, University of California, Los Angeles, California.

[2] Institute for Physical Science and Technology, University of Maryland, College Park, Maryland.

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Roegen,[1] Theil,[2] Davis,[3] Lisman,[4] and Montroll[5]). It remains a problem, in such systems, to deal quantitatively with entropy.

In information theory the entropy (or negentropy) is selected, as an information measure, by enunciating (axiomatizing) an almost exhaustive set of qualities which such a measure should have if it is to be anthropomorphically satisfying. Then it is shown that the entropy is the *unique* function which meets these requirements. The selection is further justified by its obvious utility when applications are attempted. As a result, it is a natural step, when considering entropy in a nonphysical system, to adopt a function used in information theory and to endow it with a certain primacy. As a consequence the Gibbs-like entropy function is essentially "plucked from the air" as a measure of uncertainty or disparity. Occasionally it is used in an almost mystical fashion; for example, the fractions which appear in it may not even be identified with probabilities.[2,5]

Since there are many other quantities besides the Gibbs function which could be used as measures of uncertainty, e.g., the variance, why should the entropy, among all of them, be distinguished? An answer might be found by examining how entropy arises in physics. There it may also be used as a measure of uncertainty; but as such, it arises *naturally* and is related in an *especially simple* manner to other physical measurables such as heat capacity. The same should be true in nonphysical systems; entropy should be that measure of uncertainty which is simply related to other measurables, e.g., to economic measurables such as income, profit, etc.

The focus on entropy as a *primary* concept may be misdirected, and this may be the reason for some of the controversy surrounding it. In physics, entropy is a part of thermodynamics, but it is the *entire* thermodynamic *method* which is useful and which occupies the position of primacy. This suggests that the main thrust of the search for entropy in nonphysical systems should be directed at an attempt to apply the thermodynamic *method* to these systems, whereby the entropy function will appear usefully, but incidentally! This is the direction we shall take in the present paper.

In statistical thermodynamical theory entropy usually appears during a process in which a "most probable" distribution, subject to certain constraints, is derived. However, even here, the important point to be made is that it is not the maximization of the *entropy* which is primary, but that the distribution is chosen to be completely random, except for the constraints. Entropy enters, during this process, through the "back door," and is immediately perceived as being quantitatively connected to other thermodynamic measurables. It does not, and *need not*, occupy a position of primacy.

The "maximum entropy formalism" pioneered by Jaynes,[6] and elaborated by many others,[(7a),3] especially Tribus[(8)] and Levine,[(9)] is designed to handle nonphysical systems. In this formalism one considers independent events such as the tossings of a coin or the throws of a die. It is assumed that the same spectrum of n possible outcomes is available to each event, and that the probability of the ith outcome is p_i. One wishes to infer the probabilities p_i, but not necessarily by measuring the observed frequency f_i. In fact n may be such a large number that it may be impractical to attempt a direct measurement of all the f_i. Instead it may be feasible to measure the first moments of certain quantities which are functions of the outcome i. Thus, for the rth such quantity the moment A_r is

$$A_r = \sum_{i=1}^{n} p_i A_{ri} \tag{1}$$

where A_{ri} is the value of the rth quantity associated with the ith outcome. Now we may know m first moments (i.e., r runs from 0 to $m-1$, with $m<n$). In this case, the set of equations of the type of Eq. (1) cannot be inverted to provide a unique specification of the p_i, and some algorithm is necessary to allow *some* particular inversion. The algorithm suggested by Jaynes involves maximizing the "missing information," i.e., choosing the p_i so that the missing information is maximized.

Without going into detail, since Jaynes' work is easily available for reference, the "missing information" is, as the name implies, an information theoretic concept, and really corresponds to the assumption of maximum randomness, limited only by whatever information (constraints) is already available to the observer. Such information might be represented by conditions of the type of Eq. (1).

2. THE MAXIMUM ENTROPY FORMALISM AND THE CONSISTENT INFERENCE OF PROBABILITIES

As indicated in the previous section, there are other measures of uncertainty besides entropy, and this raises the question as to why entropy, among all of them, should be distinguished. The simple relationship of entropy to other measurables furnishes one reason. However, during the past few years Levine and his coworkers[(10)] have provided another compelling reason, involving "consistency." In order to define "consistency" we again consider an event with n possible outcomes such that the probability

[3] For a review of the maximum entropy principle with emphasis on its extension to non-equilibrium statistical mechanics and irreversible processes see Ref. 7b.

of the ith outcome is p_i. Again there may be an rth quantity associated with the outcome, such that A_{ri} is the value of the quantity if the ith outcome is realized. Then, as in Eq. (1), the first moment of the average value of the rth quantity is given by A_r. The first moments of m quantities may be known. The entropy corresponding to this event is then specified by

$$S_p = -\sum_{i=1}^{n} p_i^* \ln p_i^* \tag{2}$$

in which we have used p_i^* in place of p_i. The star indicates that, in Eq. (2), the p_i^* are to be regarded as approximations to the true, fundamental probabilities p_i. Then the algorithm in the maximum entropy formalism consists of varying the p_i^*, always subject to the constraints implicit in the known m first moments of the type of Eq. (1), until the entropy S_p is maximized. Although the complete set of p_i^* obtained in this manner are considered approximate, they could in rare circumstances be identical with p_i.

The event having n possible outcomes, could be the realization of some ith possible value of some measured quantity in an experiment. Now consider a *sequence* of such measurements in which the ith outcome is realized $N_i^{(d)}$ times. If in the sequence there are N measurements, then

$$\sum_{i=1}^{n} N_i^{(d)} = N \tag{3}$$

The particular sequence is denoted by the symbol d, and by the word "sequence" we mean a series of measurements in which the various outcomes, $N_i^{(d)}$ of kind i, occur in a definite order. We can equally well define a "distribution," (denoted by the symbol D) in which the ith outcome occurs $N_i^{(D)}$ times *without regard to order*. If the events (or measurements) are independent, and the true fundamental probabilities p_i are known, then the probability P_D of the distribution D must be related to the p_i by the relation

$$P_D = \Omega_D \prod_{i=1}^{n} p_i^{N_i^{(D)}} \tag{4}$$

where

$$\sum_{i=1}^{n} N_i^{(D)} = N \tag{5}$$

and

$$\Omega_D = \frac{N!}{\prod_{i=1}^{n} N_i^{(D)}!} \tag{6}$$

The quantity Ω_D represents the number of different sequences in the distribution D.

There will also be an average value of the rth quantity associated with the distribution D. This average value will be

$$A_r^{(D)} = \left[\sum_{i=1}^{n} N_i^{(D)} A_{ri} \right] \Big/ N \tag{7}$$

Furthermore, when the p_i are the true fundamental probabilities and P_D is specified by Eq. (4) it may also be shown that

$$A_r = \sum_D P_D A_r^{(D)} \tag{8}$$

if p_i is identified with the "frequency" $f_i = \langle N_i \rangle / N$, where $\langle N_i \rangle$ is the value of $N_i^{(D)}$ averaged over the distributions. Equation (8) is the analog, for the "distributions", of Eq. (1) for the "events."

We can also define an entropy for the distributions. Thus we write

$$S_P = -\sum_D P_D^* \ln(P_D^*/\Omega_D) \tag{9}$$

where again the stars indicate that the probabilities P_D^* may be approximate. Again, the maximum entropy formalism consists of varying the P_D^* in Eq. (9) so as to maximize S_P subject to the constraints of the type of Eq. (8). This allows one to specify the entire set of probabilities P_D^*. The moments A_r may be obtained with a high degree of precision by repeated measurements, and both the probabilties p_i^* and P_D^* may be obtained by applying the maximum entropy formalism to the entropies S_p and S_P defined in Eqs. (2) and (9), respectively. For the unstarred quantities the relation between P_D and p_i is given by Eq. (4). We may now ask what the relation is between the starred quantities obtained by the respective application of the maximum entropy formalism. The (remarkable) answer contained in the proof of Levine and his coworkers[10] is that the starred quantities are also related by Eq. (4), provided that these quantities have been obtained by means of the maximum entropy principle, p_i is identified with f_i, and that the events are *independent*. Moreover, these authors show that the maximum entropy principle provides the *unique* and *only* algorithm for arriving at probability distributions (albeit approximate ones) which satisfy Eq. (4), the relation known to be true for exact probabilities. The approximate probabilities are said to be "consistent" if they satisfy Eq. (4). In this sense the maximum entropy formalism is the only algorithm which preserves "consistency."

Furthermore, Levine and coworkers show that the same probability distributions p_i^* and P_D^* can be obtained by merely invoking the constraints and requiring the distributions to be consistent in the above sense. Thus entropy does not have to be defined and *nothing* has to be maximized in order to solve the inversion problem, and generate the distributions. Again, entropy appears to be a secondary rather than a primary quantity, although, as a function, it retains its usefulness. All this distinguishes entropy from other measures of uncertainty.

Before proceeding it should be pointed out that, in what appears to be a closely related paper, Shore and Johnson[11] have axiomatized the desired properties of *inference methods* (so that they are consistent) rather than the desired properties of *information measures*, and they are able to demonstrate the uniqueness of the maximum entropy principle in the sense that "deductions made from any other information measure, if carried far enough, will eventually lead to contradictions."

The published work of Levine and his coworkers has involved a somewhat formal presentation, and it is possible that more pragmatic workers, interested primarily in applications, may not have fully appreciated its importance. Therefore, because of what we believe to be its instructional merit, we present, here, a sketch of a less rigorous method, limited among other things to cases in which $N \to \infty$, (developed by the present authors) which apparently involves the same consistency requirement, clothed in another guise. We begin by noting that P_D can be expressed as

$$P_D = \Omega_D \Big/ \sum_{D'} \Omega_{D'} \tag{10}$$

where the sum in the denominator goes over all *allowed* distributions. The use of the term "allowed" implies that some distributions may be *disallowed* by the constraints imposed on the system. When the P_D are the starred, approximate probabilities, obtained by use of an algorithm, we maintain "consistency" by continuing to use Eq. (10),

$$P_D^* = \Omega_D \Big/ \sum_{D'} \Omega_{D'} \tag{11}$$

The P_D^* are now given by Eq. (4) in which the small p_i's are starred, i.e.,

$$P_D^* = \Omega_D \prod_{i=1}^{n} (p_i^*)^{N_i^{(D)}} \tag{12}$$

Equation (11) may be transformed to

$$P_D^* = K_0 \Omega_D \tag{13}$$

where $K_0 = (\sum \Omega_{D'})^{-1}$ is a unique constant depending on constraints, but not on the individual $N_i^{(D)}$. The sum in K_0 goes only over those distributions which are allowed by the constraints.

We now replace p_i^*, in Eq. (12), by

$$p_i^* = N_i^{(D)}/N = f_i^{(D)} \tag{14}$$

in which $f_i^{(D)}$ represents the frequency distribution in the Dth distribution. This will mean that the frequency in the distribution, ultimately selected by the consistency condition, is to be identified with the probability. Since, for independent events, it can be shown that the most probable and the average distributions are identical,[10b] $f_i^{(D)}$ will also be the average frequency. The consistency condition is now implemented by equating the right sides of Eqs. (12) and (13). Taking the logarithm of both sides of the resulting equation gives

$$\sum_i^n N_i^{(D)} \ln p_i^* = \ln K_0 = K \tag{15}$$

where K is another unique constant. Substituting Eq. (14) into Eq. (15) gives

$$\begin{aligned}\sum_{i=1}^n N_i^{(D)} \ln(N_i^{(D)}/N) &= \sum_{i=1}^n N_i^{(D)} \ln N_i^{(D)} - \sum_{i=1}^n N_i^{(D)} \ln N \\ &= \sum_{i=1}^n N_i^{(D)} \ln N_i^{(D)} - N \ln N = K \end{aligned} \tag{16}$$

where we have used $\sum N_i^{(D)} = N$.

If we take the total differential of Eq. (16) with respect to the various $N_i^{(D)}$, we must bear in mind that $N_i^{(D)}$ are constrained by constraints of the type of Eq. (1), i.e., by constraints of the form

$$A_r = \sum_{i=1}^n A_{ri} p_i^* = \sum_{i=1}^n A_{ri} N_i^{(D)}/N \tag{17}$$

The differential of Eq. (16) is

$$\begin{aligned}&\sum_{i=1}^n \ln N_i^{(D)}\, dN_i^{(D)} + \sum_{i=1}^n dN_i^{(D)} - d(N \ln N) \\ &\quad = \sum_{i=1}^n \ln N_i^{(D)}\, dN_i^{(D)} + dN - d(N \ln N) \\ &\quad = \sum_{i=1}^n \ln N_i^{(D)}\, dN_i^{(D)} + 0 - 0 = 0 \end{aligned} \tag{18}$$

where we have used the facts that N and K are unique constants. Thus we arrive at

$$\sum_{i=1}^{n} \ln N_i^{(D)}\, dN_i^{(D)} = 0 \tag{19}$$

where the variations are subject to the constraints, Eq. (17). In this form the procedure becomes *identical* to the maximum entropy formalism and the constraints can be applied to Eq. (19) through the use of undetermined multipliers. Thus the derived $N_i^{(D)}$ will be exactly the same as those derived by the maximum entropy method, however, it must be remembered that the 0 in Eq. (19) comes not from a process of *maximization*, but rather from the *consistency* condition, i.e., from the fact that K is a unique constant. *Nothing* has been maximized and the entropy *never* has to be defined!

The ultimate p_i^* are then given by Eq. (14) and agree with both the maximum entropy principle and with the consistency condition, enunciated in Eqs. (12) and (13). Note that this argument directly shows that the p_i^* satisfy an exponential form. Furthermore, this solution is unique. A bit of linear algebra suffices to show that one can derive the unique expansion of $\ln p_i$, in terms of the A_{ri} from the constraints of Eq. (17) (if they are linearly independent), and that this expansion is identical to the logarithm of the p_i^*.

Again, the main point of this section is that although the maximum entropy formalism is a convenient tool, the *same* inversion can be achieved without it, by using, instead, the more satisfactory (and less subjective) consistency condition. Nothing has to be maximized, and entropy does not have to be defined. Entropy may be useful, but it is not a primary quantity.

3. DEPENDENT SUBSYSTEMS

The discussion in the preceding section is concerned with *independent* events. If we are dealing with physical systems the event may represent a subsystem (e.g., a molecule) and the "outcome," a given state of the molecule. In particular with such events or subsystems, the order in a particular sequence does not influence its probability; only the number of times that a given outcome or state occurs has influence.

For dependent subsystems, the proofs of the last section do not necessarily hold. At least, no convincing proof has thus far been advanced. Even so this does not necessarily invalidate the maximum entropy formalism, it merely forces reliance on theories of chaos and quasiergodicity alone. Nevertheless, in this paper we shall deal with an example involving only independent subsystems. Furthermore, as we shall see later, at least for one-dimensional systems, this does not necessarily mean *non interacting* systems.

4. UNDETERMINED MULTIPLIERS AS THERMODYNAMIC-LIKE VARIABLES

The approach employed in the present paper focuses on the consistent inference of probability distributions for nonphysical systems by using the maximum entropy formalism. For this purpose we need not define the entropy, but shall merely maximize Ω_D introduced in Section 2. We now know that, for independent subsystems, this is merely a convenient way to achieve consistent inference. Entropy will ultimately appear, simply connected to other measurables, but it need not be treated as a primary quantity.

In the process of extremalization certain Lagrange (undetermined) multipliers will appear. Rather than viewing these as part of a convenient device for performing the maximization we inquire into the significance of the multipliers. We discover, not surprisingly, that they are thermodynamic-like parameters, which transform by the usual methods of partial differentiation. However, in attempting to interpret them, we also realize that they summarize, in special ways, the "collective" behavior of the nonphysical system, and are therefore useful in their own rights.

Although thermodynamics or probability distributions are not mentioned by them, this last point has been made, recently, by Baxley and Moorhouse in connection with optimization in an economic problem.[12] The present paper therefore subscribes to their point of view.

When suitably interpreted, the Lagrange parameters are *natural* summarizers of collective behavior because they are simply connected, not only to one another, but also to other measurables, possibly nonphysical. They therefore form the basis of a "thermodynamics" of nonphysical systems. Even if the formalism does not lead to *exact* results, the relations can have value in a limiting sense. In the following sections we shall subject a particular system, namely, a single line of vehicular traffic, to such a thermodynamic-like treatment.

5. A SINGLE LANE OF CARS

We address a somewhat idealized version of a single lane of cars. We choose this example only because it is useful for illustrating the thermodynamic method in connection with a nonphysical system. Thus, even though we are able to arrive at some interesting and suggestive results, our goal is *not* to provide a primer on traffic engineering.

Considerable effort, both experimental and analytical, has been devoted to the study of single-lane traffic.[13-16],[4] Much of this effort has been concerned with the development of the correct "dynamical" equation (or equations) of motion and, in particular, with problems of mode structure and stability of the system.[18,19] Some work has been concentrated on the so-called "follow-the-leader" concept in which a "following" car continually accelerates in a manner which depends upon its own velocity, the velocity of the car directly in front of it (the "leader"), its distance from the leader, and the response time of the "follower." In this model a car "interacts" only with its nearest neighbor in front. We shall elaborate on this later.

Another approach[20,21] has involved studying the relation between the local linear density of cars and the local flux density, or local average velocity. In this approach it is assumed that the local average velocity is determined by the flux density alone. However, we note, for future reference in our own development, that it is possible to constrain the system in the same way that virtual variations are "constrained" to occur in thermodynamics, so that additional independent variables besides local density come into being. Thus we shall deal with situations in which average velocity depends on other variables besides local density.

Perhaps the most important distinction to be made between our development and the preceding ones concerning single-lane traffic is the fact that, in our case, the lane, even though moving, is treated as an *analog thermodynamic* system, and is therefore, in this sense, in *equilibrium.* The earlier studies deal with the problems of *nonequilibrium* transport in the system, and are often *hydrodynamic* in character. In this connection, attention is directed to the work of Prigogine and Herman,[22] who have developed a formalism, which resembles the Boltzmann transport equation of molecular kinetic theory, for treating the flow of traffic (not necessarily single lane). The important distinction of our method should be borne clearly in mind.

In the treatments of single-lane traffic undertaken previously the system is *strictly* single lane. No influences other than those internal to the system are allowed. Interaction is between cars *in* the system (follow-the-leader). Boundary conditions, e.g., the behavior of the *first* car, are also always *within* the system. Cars cannot be added laterally to the system, i.e., there is nothing which resembles lane switching. In contrast (except for lane switching), in our treatment, there can be continuous influences from outside the system, but such influences are random, i.e., the world outside of the system behaves like a "thermostat." For example, a driver can notice

[4] For overviews of some of the mathematical aspects see Ref. 17.

something to his side, and alter his velocity because of it. Thus our single lane could even be part of a multilane system, the other lanes playing the role of a thermostat. The characteristics of the "thermostat" must however be *measured* in each case, usually by determining the magnitudes of the thermodynamic-like variables (e.g., undetermined multipliers) of the system. However, as indicated, in our system, lane switching will also be forbidden. In thermodynamic terms, our system is "closed."

Some of the results, to be discussed later, of "follow-the-leader" studies indicate that, to a high degree of approximation, the velocity of the following car depends only on its distance (headway) behind the leading car. This dependence may not be absolutely determinate, for example there may be a probability distribution for the velocity in which the headway is a parameter. In any event, in this approximation, the state of the following car is determined by *its* headway and *its* velocity. Thus the system can be described by *independent* state occupation numbers as in Section 2 of this paper. The consistency arguments of that section are therefore applicable. However, we note that this does not mean that the cars are not interacting! The fact that the velocity of a following car depends on its distance from the preceding car corresponds to an "interaction" between cars.

6. THERMODYNAMIC FORMALISM FOR THE SINGLE LANE

We consider a single line of N cars, such that the distance L between the first car and the Nth is constrained to be constant. We consider a very large system so that N and L are essentially infinite while the ratio N/L remains finite. Since N and L are fixed, the linear density of cars N/L, is also fixed. We will also assume that the average velocity of a car in the system is fixed. These various conditions, of course, place constraints on the distribution of velocities (and separations) of the cars.

A few words concerning the definition of this distribution are in order. Ordinarily we would think of the set of allowable velocities and separations as forming continua. However, when we eventually deal with *entropy*, this poses a problem because the infinite number of states in the continuum causes an infinite entropy. The same problem appears in physical systems, when they are considered classically, but disappears when their ultimate quantal natures are admitted. Then the classical picture can be patched up by assigning (via the uncertainty principle) a phase space volume h^f (where h is Planck's constant and f represents the number of degrees of freedom of the system) to each "classical" state of the system.[23]

In our development we shall simplify the exposition by arbitrarily quantizing both the velocity and distance. Thus a car may have a velocity

$$v_n = nu \tag{20}$$

where n is an integer and u is the quantum of velocity. Similarly the headway for a car may be

$$l_k = kw \tag{21}$$

where k is an integer and w is the quantum of distance. Once the formalism has been developed (on a quantized basis) we shall pass to the continuum. The majority of the derived relationships (with the exception of entropy) become insensitive to the size of the quanta while the size is still finite. Thus, we can retain the quantized description without having to be too precise about the magnitudes of u and w.

Adopting the quantized approach, we define $m_{nk}uw$ as the number of cars having velocities in the range, nu to $(n+1)u$ and headways in the range, kw to $(k+1)w$. Thus m_{nk} is a density having the units of reciprocal velocity times reciprocal length. In conformance with our earlier discussion, n and k are not entirely independent. We express this fact by introducing a degeneracy $\chi_k(n)u$ which measures the number of states having velocities in the range, nu to $(n+1)u$, *when the headway is kw*. With this definition we can conveniently express the constraints of constant N, constant L, and constant average velocity as follows:

$$\sum_n \sum_k m_{nk} uw = N \tag{22}$$

$$\sum_n \sum_k m_{nk} uw(kw) = L \tag{23}$$

$$\sum_n \sum_k m_{nk} uw(nu) = N\bar{v} \tag{24}$$

in which $\bar{v}$ is the constant average velocity. The sums in Eqs. (22), (23), and (24) *are understood to include* the restriction implicit in the dependence of n upon k. Assuming that all sequences of cars (allowed by the constraints) are equally probable we may apply the maximum entropy formalism and maximize

$$N! \Big/ \prod_n \prod_k (m_{nk} uw)! \tag{25}$$

subject to the constraints, Eqs. (22) through (24). Employing the method of undetermined multipliers in the usual manner, we find

$$m_{nk} uw = (N/\Delta) e^{-\alpha kw} e^{-\beta nu} \tag{26}$$

in which Δ is given by

$$\Delta = \sum_n \sum_k e^{-\alpha kw} e^{-\beta nu} \tag{27}$$

and where α and β are undetermined multipliers which have to be determined by the substitution of Eq. (26) into Eqs. (23) and (24). Equation (26) can be used as a basis for developing a thermodynamics of the traffic system, defining the probability $P_{nk}uw$ as

$$P_{nk}uw = m_{nk}uw/N \tag{28}$$

whereupon Eq. (26) may be expressed as

$$P_{nk}uw = e^{-\alpha kw} e^{-\beta nu}/\Delta \tag{29}$$

We define the quantities,

$$T = 1/\beta \tag{30}$$

$$p = \alpha/\beta \tag{31}$$

$$\bar{s} = -\sum_n \sum_k (P_{nk}uw) \ln(P_{nk}uw) \tag{32}$$

whence it is easily shown that

$$\bar{v} = T\bar{s} - T \ln \Delta - p\bar{l} \tag{33}$$

where

$$\bar{l} = L/N \tag{34}$$

Maintaining N constant, and regarding α and β as the independent variables, we easily find from the definition of Δ in Eq. (27), and the definitions of $\bar{v}$ and $\bar{l}$ in Eqs. (24) and (34), that

$$d \ln \Delta = -\bar{l}\, d\alpha - \bar{v}\, d\beta \tag{35}$$

Furthermore it follows from Eqs. (30) and (31) that

$$dT = -(1/\beta^2)\, d\beta \tag{36}$$

$$dp = -(1/\beta)\, d\alpha - (\alpha/\beta^2)\, d\beta \tag{37}$$

Finally, from Eq. (33), we find

$$\ln \Delta = \bar{s} - (\bar{v}/T) - (p\bar{l}/T) \tag{38}$$

Taking the total differential of $\bar{v}$ in Eq. (33), and using Eqs. (35) through (38) yields

$$d\bar{v} = T\, d\bar{s} - p\, d\bar{l} \tag{39}$$

This equation looks suspiciously like the equation based on the combined first and second laws of thermodynamics[24] in which $\bar{v}$ is the *analog* of the internal energy per molecule, $\bar{s}$ is the *analog* of the entropy per molecule, and $\bar{l}$ is the analog of the volume per molecule, while T and p are the analogs of the temperature and pressure of a molecular system, respectively. We will of course adopt this correspondence, and call T and p the temperature and pressure. Since we are dealing with a one-dimensional system our "volume," $\bar{l}$, is actually a length. Reference to Eq. (32) shows that $\bar{s}$ indeed has the usual form of the entropy function and is therefore, appropriately the entropy. Unlike the other thermodynamic analogs, it is dimensionless. Furthermore, it has not merely been "plucked from the air," but has arisen naturally, and is simply connected to other traffic measurables such as $\bar{v}$ and $\bar{l}$.

The quantity Δ, as we continue to develop the analogy, plays the role of a partition function in the analog statistical thermodynamics. We note, from Eq. (38), that

$$\bar{v} - T\bar{s} + p\bar{l} = -T \ln \Delta = \bar{g} \tag{40}$$

where we have symbolized the sum on the left by the quantity $\bar{g}$. By comparison with the well-known thermodynamic function, we recognize that $\bar{g}$ is the analog of the Gibbs free energy for a physical system, and its relation to Δ in Eq. (40) identifies it as the *characteristic function* for the partition function which Δ represents

$$\Delta = \sum_n \sum_k e^{-pkw/T} e^{-nu/T} \tag{41}$$

which, by comparison with its physical counterpart, is clearly the partition function in the constant pressure ensemble.[25] Furthermore, it is well established that the characteristic function for the partition function in this ensemble is the Gibbs free energy. Thus the analogy is (not surprisingly) complete.

At this point it is convenient to remember that the sum $\sum\sum$ in Eq. (41) is restricted to terms allowed by the dependence of n upon k. In fact we can symbolize this restriction by writing

$$\begin{aligned}\Delta &= \sum_k e^{-pkw/T} \left[\sum_{n(k)} e^{-n(k)u/T} \right] \\ &= \sum_k e^{-pkw/T} \left[\sum_n \chi_k(n)\, u e^{-n(k)u/T} \right] \\ &= \sum_k e^{-pkw/T} q(k, T) \end{aligned} \tag{42}$$

where $\chi_k(n)\,u$ is the previously mentioned degeneracy, and

$$q(k, T) = \sum_n \chi_k(n)\, u e^{-n(k)u/T} \tag{43}$$

is obviously the analog of the partition function in the canonical ensemble.[23]

Of course, the Boltzmann constant does not appear in our development, since it obviously has no meaning in the present context. We could, of course, imitate the Boltzmann constant by redefining the temperature scale so that a constant would appear in front of T in Eq. (41). This, however, would be pure definition, and so we avoid it. However, in any given system α and β or, alternatively, p and T, must be determined from the experimental data. We address this subject later.

7. INTERPRETATION OF THERMODYNAMIC-LIKE QUANTITIES

Beginning with Eq. (39), the analogs of other thermodynamic quantities, besides T, p, $\bar{s}$, and $\bar{l}$, can be defined, and the relation between them determined by means of partial differentiation, just as such relations are developed in ordinary thermodynamics. As indicated earlier, in connection with the discussion of undetermined multipliers, the significances of the analog quantitites need to be investigated.

The entropy $\bar{s}$ retains its usual significance as a measure of disorder. How about the temperature? From Eq. (39) we see that

$$T = (\partial \bar{v}/\partial \bar{s})_{\bar{l}} \tag{44}$$

We would expect that (at typical velocities) as the average velocity $\bar{v}$ is increased, the drivers would, on the basis of safety requirements, not tolerate much disorder. Thus, at typical velocities, we would except that $\bar{s}$ would be a decreasing function of $\bar{v}$. At very high average velocities we would expect the drivers to be organized to the extent that they would all drive with about the same velocity.

If, as indicated, $\bar{s}$ is a decreasing function of $\bar{v}$, the derivative in Eq. (44) would be negative. Thus we expect the "traffic temperature," T, to be *negative*. We shall see, later, that, at typical velocities, the experimental data require T to, indeed, be negative.

Are there any circumstances under which we should expect T to be *positive*? Consider the case of a traffic jam when all vehicles are stopped and very close to one another. As the jam begins to break, the cars *increase* their velocities, $\bar{v}$ and $\bar{l}$ remain initially small, and we expect the degree of disorder to *also* increase, i.e., the distributions of spacings and velocities should *both* broaden. Under these conditions $\bar{s}$ *increases* with $\bar{v}$, and Eq. (44) requires T to be *positive*. This conjecture is also confirmed later.

T summarizes the "collective" behavior of the drivers, and must be determined from experimental data. As such it is one of the convenient parameters which characterizes those data. Its modulus admits of a nice interpretation. The derivative on the right of Eq. (44) identifies T as the rate of change of the collective average velocity with the degree of collective disorder. It represents the collective response of the system to a possibly threatening change in the degree of disorder. Thus $|T|$ (the absolute value being chosen to assure a positive quantity) might be viewed as the system's collective sensitivity to a change in degree of order (disorder). We shall henceforth refer to $|T|$ as the *collective sensitivity* of the single-lane system. Conceivably, it could be a parameter useful to traffic engineers. However we do not pursue this point here.

Similar considerations can be advanced with respect to p, the traffic pressure, which, according to Eq. (39), may be represented as

$$p = -(\partial \bar{v}/\partial \bar{l})_{\bar{s}} \tag{45}$$

The traffic pressure is therefore the rate of change of average velocity with respect to the average spacing, when the degree of order is maintained constant. One would expect that, at typical velocities, the drivers would slow down when the spacing between cars is decreased, so that when the denominator in the derivative on the right of Eq. (45) is negative the same would be true of the numerator, and the derivative itself would be positive. Equation (45) then requires that p itself be negative. This conclusion can be verified with more rigor by referring to Eq. (41) where the partition function Δ is represented by the sum on the right. If T is negative then this sum would not converge (since it extends to infinite values of n) unless p were negative.

The coefficients $(\partial \bar{s}/\partial \bar{l})_T$, $(\partial \bar{s}/\partial p)_T$ are also of interest. However, they are related, through Maxwell relations,[24] to derivatives which can be obtained from the equation of state which we discuss later. We therefore delay discussion of these derivatives until then.

As an example of the usefulness of thermodynamic-like variables, consider the flux of f of cars. This is given by the product of the average density and the average velocity. Thus

$$f = \bar{v}/\bar{l} \tag{46}$$

One might be interested in setting a speed limit (or doing something) to control the average velocity in such a way that f is maximized subject to the maintenance of a good level of "collective sensitivity," $|T|$, because such a level might help in the avoidance of accidents. Then one would be interested in the derivative

$$(\partial f/\partial \bar{v})_T \tag{47}$$

In order to evaluate this derivative it would be convenient to express f as a function of $\bar{v}$ and T. This can obviously be accomplished by using the partition function to evaluate $\bar{v}$ and $\bar{l}$ as functions of T and p, and, then, through Eq. (46), f as a function of these variables. Then one could use standard thermodynamic transformation theory, involving partial differentiation, to obtain the derivative in Eq. (47).

8. "FOLLOW-THE-LEADER" INTERACTION

As indicated earlier, the subject of the appropriate theory for modeling cars in a single lane of traffic has had extensive study, and, one approach, the so-called "follow-the-leader" concept, has received considerable attention.[5] Several empirical studies pursued from different points of view, including some theoretical considerations, have suggested a common form for the interaction between cars. We shall adopt this form. However, it should be emphasized, once more, that the thermodynamic approach is in no way limited to this particular form. We choose it because it has been verified by some investigators, and because it is particularly convenient for illustrating the theory.

Several idealizations are introduced at the outset. All the cars in the single lane are assumed to be identical, and all the drivers are supposed to have the same response. Clearly we can only be talking about some sort of average car, and also about an average driver. The interaction is characterized by having the acceleration of the jth car, in a line of cars, expressed in terms of the velocities and positions of both the $(j-1)$st and jth car. This relation is

$$\frac{dv_j(t+\bar{\tau})}{dt} = \lambda_0 \left[\frac{v_{j-1}(t) - v_j(t)}{x_{j-1}(t) - x_j(t)} \right] \tag{48}$$

In this equation v_j is the velocity of the jth car and x_j is its position, while v_{j-1} and x_{j-1} are the corresponding quantities for the $(j-1)$st car. The quantity t represents time, and it is evident from Eq. (48) that v_j and x_j are

[5] For reviews of car-following, hydrodynamic, and kinematic models see, for example, Refs. 16, 17, and 26. Single-lane traffic models and experiments have been addressed in a series of symposia on traffic flow theory; for example see Refs. 14 and 27.

regarded as functions of time. The quantity $\bar{\tau}$ in Eq. (48) is a "response time," and its appearance in the equation indicates that the acceleration, dv_j/dt, which the jth car undertakes, at the time $t+\bar{\tau}$, is in response to a stimulus which occurred (right side of the equation) at time t. The parameter λ_0 is called the sensitivity,[6] and may vary from road to road. For example,[15] the values of λ_0 measured, respectively, on the General Motors test track, in the Holland Tunnel, and in the Lincoln Tunnel are 27.4, 18.2, and 20.3 miles per hour. Incidentially, the values of the response time measured on these respective roads are 1.5, 1.4, and 1.2 sec. We note that v_j appearing in Eq. (48), is given by

$$v_j = dx_j/dt \tag{49}$$

Apart from the effect of the response time, Eq. (48) indicates that the acceleration of the following car is proportional to the difference in velocities between the two cars (leading and following), and inversely proportional to their separation. This inverse relation provides a damping factor, so that when the cars are separated by a large enough distance there is effectively no interaction. Equation (48) can *almost* be arrived at without experiment if one demands that the interaction between the cars be of such a nature that a disturbance, generated by the erratic behavior of some car in the line, *not* be propagated along the line as a growing wave, and an instability.[15]

Other relations resembling Eq. (48) have been suggested (see Ref. 16). For example, some of them raise the numerator and denominator in the brackets of Eq. (48) to various powers. Others introduce functions between λ_0, and the bracketed expression. Some variations include the physical effects of inertia. However, in the end, these variations do not introduce dramatic changes into the overall behavior of the traffic system.

Equation (48) can only have a probabilistic meaning. Thus given $v_{j-1}-v_j$ and $x_{j-1}-x_j$, a given acceleration is likely to be observed with a well-defined probability. The acceleration on the right of Eq. (48) should really be interpreted as some *average* acceleration. In order to achieve a useful simplication, in Eq. (48) this average has been replaced by the actual acceleration in which v_j on the left is identical with v_j on the right. Clearly, serious order-of-averaging effects may have been ignored.

The dynamics of the line of cars is of course determined by Eq. (48). This is a nonlinear relation which can lead to complicated motion. However, if one *ignores the response time* $\bar{\tau}$ (because it is relatively short—an excellent approximation in many circumstances), Eq. (48) leads

[6] In view of the fact that the absolute value of the "traffic temperature" was interpreted, in Section 7, as the "collective sensitivity" we will refer to λ_0 as the "individual sensitivity."

to an extremely simple relation between v_j and $x_{j-1} - x_j$, the distance (headway) separating the two cars. Thus, setting $\bar{\tau} = 0$, and substituting Eq. (49) into the right-hand side of Eq. (48), leads to the result

$$dv_j = \lambda_0 d \ln(x_{j-1} + x_j) - \lambda_0 d \ln l_j \tag{50}$$

in which

$$l_j = x_{j-1} - x_j \tag{51}$$

and represents the distance by which the jth car trails the $(j-1)$st car. Equation (50) can be integrated immediately, subject to the condition that v_j is 0 when $\lambda_j = a$, where a is the observed characteristic distance between the centers of the two cars when they come to a halt. It is only common sense to assume that an individual driver will bring his car to a stop when it is close to the leading car. The integration of Eq. (50) in this manner then yields the relation

$$l_j = a e^{v_j/\lambda_0} \tag{52}$$

This relation [Eq. (52)] is what, for lack of a better term, we shall call the "interaction" between the cars. It is the same for every car, and requires that a car traveling at a definite velocity (with respect to a fixed "laboratory" frame of reference) trail the preceding car by a prescribed distance.

If we had not made the possibly severe approximation in which an *average* acceleration on the left of Eq. (48) is replaced by dv_j/dt in which v_j is the v_j appearing on the right, then Eq. (52) would be replaced by

$$\text{Prob}(v_j) = f(l_j, v_j) \tag{52a}$$

where $\text{Prob}(v_j)$ is the probability of the jth car having the velocity v_j when the separation is l_j. In effect we have simplified the relation to

$$v_j = \lambda_0 \ln(l_j/a) \tag{52b}$$

where Eq. (52b) is simply another form of Eq. (52).

When working with the "quantized" system of Section 6 Eq. (52a) can be associated with the degeneracy $\chi_k(n)u$ appearing in Eqs. (42) and (43), and describes the number of states lying in the velocity range nu to $(n+1)u$ when the separation between cars is kw. The quantities u and w are the previously defined "quanta" of velocity and distance, while n and k are integers. The correspondence is perhaps clearer if we write, in place of Eq. (52a),

$$\text{Prob}_k(n)u = f(k, n)u \tag{52c}$$

Clearly, $\mathrm{Prob}_k(n)u$ and $\chi_k(n)u$ are considered to be proportional to one another. Equation (52) also introduces another important point. The quantities v_j and l_j which appear in it refer *only* to the jth car. Therefore, although Eq. (52) represents an *interaction* between cars, since l_j is a distance between cars, the *state* of the jth car is defined *independently* of the states of the other cars. Thus the "consistency" arguments of Levine and coworkers[10] apply.

One last point concerning Eq. (52) is worth mentioning. Measurements have been performed on the actual flow of traffic in a single lane, e.g., in the Holland Tunnel in New York.[13] These measurements which deal with the "collection" of cars lead to the following empirical relation between traffic flux q, (cars/second) and average linear density of cars k (cars/ft):

$$q = kc \ln(k_i/k) \tag{53}$$

where c and k_i are constants. [We use q and k for the flux and linear density, *only* in Eq. (53), because they were so used in Ref. 13. Elsewhere in the present paper q and k have different meanings.] Equation (53) is easily rationalized in terms of Eq. (52), if $q = kv$; it is in fact the same relation.

However Eq. (53) results from a direct measurement on the *collection* of cars while Eq. (52) was obtained by integration of the differential equation, Eq. (48), discovered in measurements on only *two* cars, a leader and a follower. These extreme ways of arriving at a common result provide it with a measure of respectability.

9. DEALING WITH THE DEGENERACY $\chi_k(n)u$

Unfortunately no useful data appear to exist concerning the probability distribution, $\mathrm{Prob}_k(n)u$ of Eq. (52c), in which u is a quantum of velocity, or equivalently, for the degeneracy $\chi_k(n)u$ which appears in Eq. (43) of Section 6. As a result, in order to continue the development, we rely on an inversion of the order of averaging, which we now discuss.

We define a number $\bar{n}(k)$ as the *closest* integer to $n^*(k)$ defined by

$$e^{-n^*(k)u/T} = q(k, T) \tag{54}$$

so that the degeneracy, $\chi_k(n)u$, and the summation of Eq. (43) are taken into account. Then, with minimum error (especially when u and w are small) we replace Eq. (54) with

$$e^{-\bar{n}(k)u/T} = q(k, T) \tag{55}$$

which converts Eq. (42) into

$$\Delta = \sum_k e^{-pkw/T} e^{-\bar{n}(k)u/T} \tag{56}$$

Now kw is determined as a function of $\bar{n}u$ by Eq. (55), so we can write

$$k(\bar{n})w = l_{\bar{n}} \tag{57}$$

and, in place of Eq. (56), we then get

$$\Delta = \sum_n e^{-pl_{\bar{n}}/T} e^{-\bar{n}u/T} \tag{58}$$

As $u \to 0$ we can replace this sum by an integral, writing

$$\Delta = \int_0^\infty e^{-pl_{\bar{n}}/T} e^{-\bar{n}u/T}\, d\bar{n} \tag{59}$$

Replacing $\bar{n}u$ by y converts Eq. (59) into

$$\begin{aligned} \Delta &= \left(\int_0^\infty e^{-pl(y)/T} e^{-y/T}\, dy \right) \Big/ u \\ &= \Delta_0/u \end{aligned} \tag{60}$$

We now examine the influence of the quanta u and w on the various thermodynamic variables. We first examine $\bar{s}$ in this context. The most convenient equation to start with is Eq. (38) which may be written as

$$\bar{s} = \frac{\bar{v}}{T} + \frac{p_{\bar{l}}}{T} + \ln \Delta \tag{61}$$

In this equation we know that $\bar{v}$ and $\bar{l}$ are independent of u and w since they are fixed by the constraints. However, even with very small quanta, the same cannot be said of $\ln \Delta$ in Eq. (60) where Δ_0 is the integral in the parentheses, and clearly does not depend upon u or w.

According to Eq. (60) we have

$$\ln \Delta = \ln \Delta_0 - \ln u \tag{62}$$

Thus as u goes to 0, $\ln \Delta$ becomes infinite. At the same time, since $\ln \Delta$ appears in Eq. (61), $\bar{s}$ becomes infinite. The situation is somewhat saved by the fact that we are usually interested in entropy differences, and, in that case, the $\ln u$ will cancel out of the difference.

Next we examine how T and p depend on u as it becomes small. For this purpose we rearrange Eq. (61) to read

$$T(\bar{s} - \ln \Delta) = \bar{v} + p\bar{l} \tag{63}$$

Since both $\bar{s}$ and $\ln \Delta$ contain the term $\ln u$, that term cancels out of the expression in parentheses on the left of Eq. (63). Thus if $\bar{v}$ and $\bar{l}$ do not depend on u, as we have indicated, there is no reason for either T or p to depend upon u. However, this argument is based on the assumption that u is indeed small. Just how small u must be before it no longer plays a role in the thermodynamics (outside of its influence on the entropy) is a matter which can only be determined by numerical analysis. We can be fairly confident, however, that u will cease to play a role when it reaches some level of smallness. Among other things, we examine this question later.

Up to this point our development of the single-lane analog thermodynamics only requires that the state of a following car be determined by its own parameters, i.e., by its headway and its velocity. As such, the single-lane system is characterized by independent state probabilities for each car. Furthermore the theory readily accommodates a possible dependence of velocity upon headway. Hence, the theory is quite general, and can be applied to numerous specific car-following laws.

Returning to Eq. (52), the quantized version becomes

$$kw = ae^{\bar{n}(k)u/\lambda_0}$$

or

$$\bar{n}(k)u = \lambda_0 \ln(kw/a) \tag{64}$$

Then from Eq. (55) we have for the "canonical ensemble" partition function

$$q(k, T) = (a/kw)^{\lambda_0/T} \tag{65}$$

All of these equations would be precise if the velocities corresponding to a given headway (car separation) were narrowly distributed as a delta function about a central velocity $\bar{n}(k)u$.

Making this assumption we begin our application, in the next section, by deriving an equation of state and then examining the effects of a passage to the continuum in which the "quanta" u and w achieve zero size.

10. EQUATION OF STATE AND SENSITIVITY OF THE STATE VARIABLES TO THE SIZE OF THE QUANTA

Adopting the formalism developed in Section 6 we derive the equation of state by employing the thermodynamic relation

$$\bar{l} = (\partial \bar{g}/\partial p)_T = -\{\partial(T \ln \Delta)/\partial p)\}_T = -T\{\partial(\ln \Delta)/\partial p\}_T \tag{66}$$

In the following, we delete the bar in $\bar{n}$, and assume that $\bar{n}$ or simply n is related to k by Eq. (64). In order to evaluate $\bar{l}$, according to Eq. (66), we need first to evaluate Δ. In doing this we will assume that, in the real case, the size of the quantum, u, is small enough, so that in passing to the continuum, we can use the integral version of Δ_0 appearing in Eq. (60). Later in this section we test the validity of this assumption.

In performing the integration, it is convenient to use Eq. (52) to transform from y (the velocity), in Eq. (60), to the spacing $l(y)$. With this transformation Δ_0 becomes

$$\Delta_0 = \lambda_0 a^{\lambda_0/T} \int_a^{\infty} l^{-[(\lambda_0/T)+1]} e^{-pl/T}\, dl \tag{67}$$

It is even more convenient to "scale" within the framework of a "law of corresponding states" for our traffic system, by transforming, in Eq. (67), to the following dimensionless variables:

$$\phi = ap/\lambda_0 \tag{68}$$

$$\xi = l/a \tag{69}$$

$$\tau = T/\lambda_0 \tag{70}$$

Equation (67) now becomes

$$\Delta_0 = \lambda_0 \int_1^{\infty} \xi^{-(1/\tau+1)} e^{-\phi\xi/\tau}\, d\xi \tag{71}$$

and Eq. (66) may be expressed, in reduced form, as

$$\bar{\xi} = \bar{l}/a = -\tau\{\partial(\ln \Delta)/\partial\phi\}_\tau = -\tau\{\partial(\ln \Delta_0)/\partial\phi\}_\tau \tag{72}$$

Clearly ϕ, ξ, and τ, are, respectively, the reduced pressure, spacing, and temperature.

Now, when $(1/\tau)+1 = -n$, where n is a positive integer or 0, the integral in Eq. (71) has an analytical representation so that we may express the partition function as

$$\Delta_0 = \lambda_0 e^{(n+1)\phi} \sum_{k=0}^{n} (n!/k!)[-(n+1)\phi]^{k-n-1} \tag{73}$$

in which the reduced pressure ϕ is assumed to be negative (as is the case in the Holland Tunnel data which we discuss later), and

$$n = -(1/\tau) - 1 \tag{74}$$

Substitution of Eq. (73) into Eq. (72) finally yields the reduced equation of state,

$$\bar{\xi}=1-\left\{\sum_{k=0}^{n}[n!(k-n-1)!/k!][-(n+1)\phi]^{k-n-2}\right\}\Big/\left\{\sum_{k=0}^{n}(n!/k!)[-(n+1)\phi]^{k-n-1}\right\} \tag{75}$$

in which the reduced temperature may be easily inserted by use of Eq. (74). We note that the equation of state in no way depends on the quanta u or w.

Figure 1, which exhibits plots of Eq. (75), shows three "isotherms" corresponding to the reduced temperatures $\tau=-0.125$, -0.167, and -0.250 which are, in turn, derived by assuming values $n=7$, 5, and 3, respectively. The ordinate in the figure is the reduced pressure ϕ and the abscissa, the reduced average spacing $\bar{\xi}$. The three particular reduced temperatures have been chosen because they bracket the range in which the actual traffic temperature lies in the study (to be discussed in the next section) of velocity and headway distributions in the Holland Tunnel.

The isotherms, in Fig. 1, all decrease sharply to very large negative pressures when the average spacing between cars becomes small, demonstrating, as expected, that the drivers decelerate very rapidly when

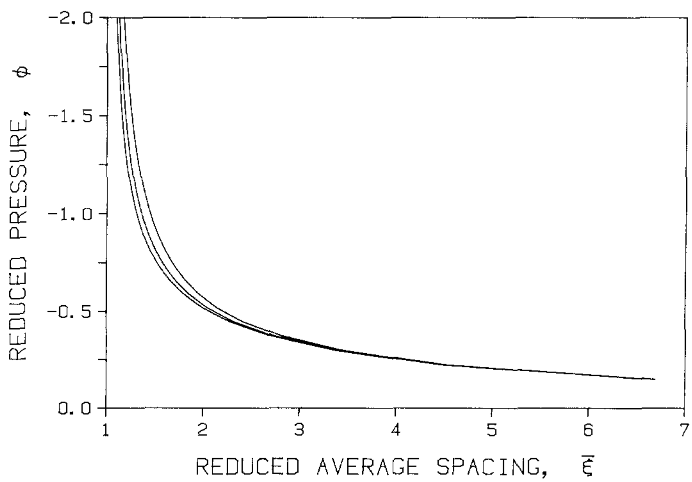

Fig. 1. Theoretical traffic isotherms. Reduced pressure, ϕ, as a function of reduced average spacing $\bar{\xi}$. Isotherms determined by the reduced equation of state, Eq. (75), corresponding to reduced temperatures, τ, of -0.125 ($n=7$), -0.167 ($n=5$), and -0.250 ($n=3$), respectively, from left to right.

they are close to one another, and, at the same time are forced to move even closer while the degree of order remains unchanged. The pressure reaches $-\infty$ only at the halting distance, $\bar{\xi}=1$. The drivers really "put on the brakes" to avoid collisions. At very large separations the isotherms converge on one another. This implies that the "collective sensitivity," which the temperature represents, plays a minimal role in the collective driving behavior when car separations are large. The dependence of entropy on spacing and pressure are of interest, and the corresponding derivatives are given by the standard "Maxwell" relations. These are

$$(\partial \bar{s}/\partial \bar{l})_T = (\partial p/\partial T)_{\bar{l}} \tag{76a}$$

$$(\partial \bar{s}/\partial p)_T = -(\partial \bar{l}/\partial T)_p \tag{76b}$$

The signs of the derivatives on the right sides of the equation may be obtained from an examination of the isotherms of Fig. 1. Thus the sign of the right side of Eq. (76a) may be determined by following the pressure parallel to the ordinate in the figure. Along such a line, the pressure increases negatively with elevation, but so does the temperature as we move from isotherm to isotherm. Thus, the right side of Eq. (76a) is positive, and so we learn, form the left side, that at constant temperature (constant "collective sensitivity") the degree of disorder increases with increasing average spacing (at least when the temperature is negative). In an exactly similar manner we find by transversing a line parallel to the abscissa (at constant ordinate), from left to right in the figure, that the right side of Eq. (76b) is also positive. Then from the left side, we learn that, at constant collective sensitivity, the degree of disorder also increases with *pressure*. This is not very intuitive, but it appears to be the case. The nonintuitive aspect of this result originates, of course, in the condition of constant collective sensitivity, i.e., constant $|T|$, a constraint whose consequences are not intuitively simple.

Another interesting feature is the *positive* slope of pressure versus spacing in Fig. 1, i.e.,

$$(\partial p/\partial \bar{l})_T > 0 \tag{77}$$

If we define the analog *Helmholtz* free energy, namely,

$$\bar{a} = \bar{v} - T\bar{s} \tag{78}$$

then, using Eq. (77) together with the usual procedure of stability theory,[24] it is possible to show that

$$(\Delta \bar{a})_{T,\bar{l}} < 0 \tag{79}$$

for a fluctuation in the local linear density, in an equilibrium state described by the equation of state in Fig. 1. Thus, *unlike* the case of a physical system, the Helmholtz free energy achieves a *maximum* (rather than a *minimum*) in the unfluctuated state. This is intuitively satisfying since one expects the drivers to attempt to maximize $\bar{v}$, their average velocity in the regression of a fluctuation, while we know that the entropy will be increased as the constraints corresponding to the fluctuation are removed. However, in this (traffic) case, T is *negative* so that an increase in $\bar{s}$ in Eq. (78) translates into an increase in $\bar{a}$, in addition to the increase due to the increase in $\bar{v}$. Thus $\bar{a}$ tends to a maximum!

We have drawn attention to the fact that, in the various dynamic theories, velocity is regarded as a function only of average density. However, as early as Eq. (39) we have exhibited $\bar{v}$ as depending on *two* variables, $\bar{s}$ as well as $\bar{l}$. How do we reconcile these points of view? The answer is as follows.

When traffic flows through the Holland Tunnel, for example, it has a particular temperature, or entropy, etc., whatever additional variable we wish to use, which has already adjusted itself, and, in the simplest case, remains constant. No attempt has been made, in the past, to measure this constant (but variable) temperature. (Later, in this paper, we do make such an attempt.) Thus there appears to be but one variable.

However, additional variables could be "constrained into action" if we so wished. For example, a long line of traffic might have *both* its density and velocity arbitrarily controlled by having the first and last cars constrained to remain a constant distance from one another, and to both move at the same fixed velocity. This constraint may be somewhat artificial, but this is often the case with "virtual variations" in thermodynamics. The values of $\bar{v}$ and $\bar{l}$ which are fixed by the constraint may *not* satisfy the simple relation connecting them in the unconstrained case. In this case the functional relationship *must* display another variable.

The integral in Eq. (71), and consequently the equation of state, Eq. (75), is derived on the assumption that the quantum, u, is small enough so that the same result is obtained, by passing to the continuum, as would be obtained if we performed the actual sum. How small must u be before this is true? We examine this question using a system in which the individual sensitivity $\lambda_0 = 27.79\ \text{ft sec}^{-1}$, the halting distance $a = 30.34$ ft, $p = -0.2090\ \text{sec}^{-1}$, and $T = -4.441\ \text{ft sec}^{-1}$. These values of the various parameters are typical of real traffic systems. For this system, the average velocity $\bar{v}$ and the average spacing $\bar{l}$, as functions of the size of the quantum (in feet per second), have been evaluated by performing the actual sums numerically in the partitition function, rather than by passing to the continuum and integrating. A result (for the average velocity) is shown in Fig. 2.

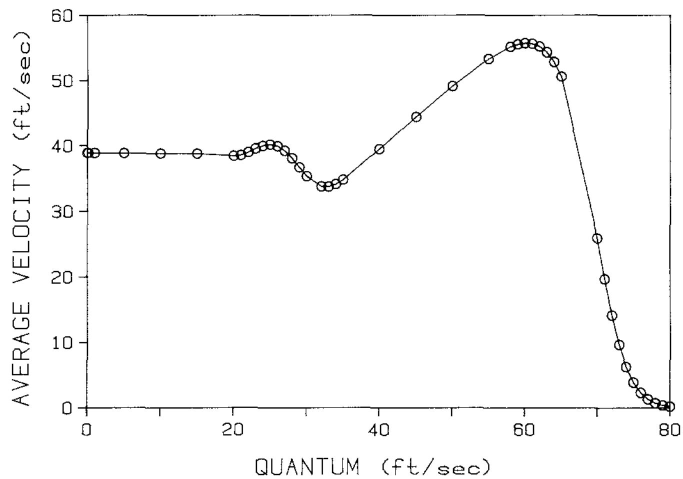

Fig. 2. Effect of quantum on average velocity. Average velocity, $\bar{v}$, versus size of the quantum of velocity, u. Parameters employed: $\lambda_0 = 27.79$ ft sec^{-1}, $a = 30.34$ ft, $\alpha = 0.047055$, and $\beta = -0.225182$ ($p = -0.2090$ sec^{-1} and $T = -4.441$ ft sec^{-1}).

In the figure, in spite of the erratic behavior of the function at larger values of the quantum, we find that the average quantity becomes constant when u is less than or equal to the *individual sensitivity*, λ_0. Thus, it appears feasible to pass the continuum if the quantum is smaller than the *individual sensitivity*.

Note that $|T|$, the collective sensitivity, and λ_0, the individual sensitivity, have the same dimensions, namely, those of velocity.

In the next section we deal with some experimental data.

11. TREATMENT OF EXPERIMENTAL DATA

In this section we attempt a comparison with existing experimental data. For this purpose we set the quantum u equal to unity. We shall be interested in systems in which λ_0 is considerably larger than unity, and, according to Fig. 2, this choice of u should place us in the range where the size of the quantum has no effect on the result. With this convenient simplification Eq. (29) becomes

$$P_v = \{\exp[-(pae^{v/\lambda_0} + v)/T]\}/\Delta_0 \tag{80}$$

in which we have used Eqs. (30), (31), (52), (57), and (60), and where we use $v = n$, i.e., v is an integer. Now, apparently, no measurements have been made, designed primarily to determine the distribution P_v, or, for that mat-

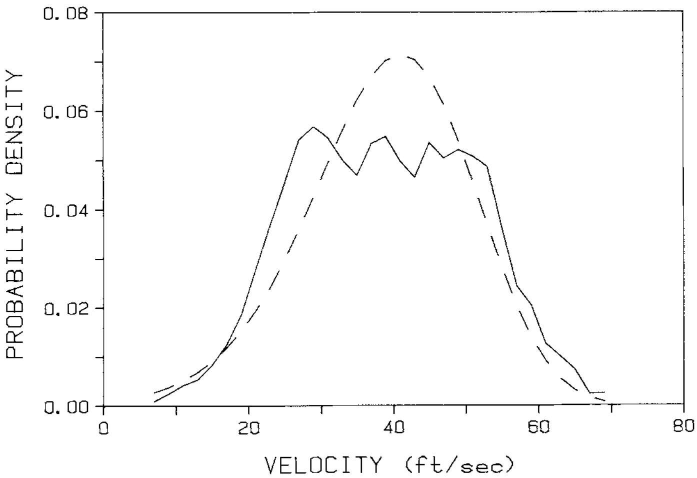

Fig. 3. Experimental and theoretical velocity distributions. Solid curve: normalized probability density as a function of velocity determined from the Holland Tunnel traffic data of Edie *et al.* Dashed curve: normalized probability density, P_v of Eq. (80), where $\lambda_0 = 27.79$ ft sec^{-1}, $a = 30.34$ ft, $p = -0.2090$ sec^{-1}, and $T = -4.441$ ft sec^{-1}.

ter, the distribution of spacings between cars. However, there is a paper by Edie, Foote, Herman, and Rothery which describes a study in which the primary purpose was to measure both the average velocity and average flux density of cars as a function of the average headway. However, in order to acquire this information, the authors had to perform some measurement of the above-mentioned distribution. Thus, inadvertently, some appropriate experimental data are available. The measurements were made in the Holland Tunnel in New York. One of the goals of the study was to validate the relation in Eq. (52), and, indeed, a reasonable experimental verification was accomplished.

The study involved 23,377 cars. Table II of Edie *et al.* reports the data. In the table the cars are classified into velocity intervals of 2 ft/sec. The observed number of cars in each such velocity interval is listed in the table, as well as the average headway of these cars. The solid curve in Fig. 3 is the normalized probability density per unit velocity [corresponding to P_v in Eq. (80)] obtained from these data. The jagged section of the curve, centered on 40 ft sec^{-1}, seems to indicate that, even though more than 23,000 cars were involved, this number was still not large enough to average all the noise. Alternatively the rather structured fluctuations might persist in an even larger sample, and indicate that the system is not quite ergodic. Only further work, both theoretical and experimental (further observation of a larger number of cars) can resolve this question. Nevertheless, assuming that the system is *almost* ergodic, we can compare the observed data with the theoretical prediction of Eq. (80). For this purpose we need the parameters a, λ_0, T, and p. The quantities a and λ_0 were measured directly (from the data) by the authors. They report

$$a = 30.34 \text{ ft}$$

$$\lambda_0 = 27.79 \text{ ft sec}^{-1} \tag{81}$$

The quantities T and p can also be determined from their data. We note that both the average spacing $\bar{l}$ and the average velocity $\bar{v}$ can be obtained by weighting the various spacings and velocities with the probability P_v specified by Eq. (80). Actually, from the definition of Δ, it is easy to show that this result corresponds to the relations

$$\bar{l} = -T\{\partial \ln \Delta(p, T)/\partial p\}_T \tag{82}$$

$$\bar{v} = T^2\{\partial \ln \Delta(p, T)/\partial T\}_p + pT\{\partial \ln \Delta(p, T)/\partial p\}_T \tag{83}$$

Thus, if $\bar{l}$ and $\bar{v}$ are available from the observed data, Eqs. (82) and (83) can be solved for T and p. We have evaluated $\bar{l}$ and $\bar{v}$ from the data in Table II of Edie *et al.*, and used Eqs. (82) and (83) for the determination of T and p. The values of these parameters are

$$T = -4.441 \text{ ft sec}^{-1} \tag{84}$$

$$p = -0.2090 \text{ sec}^{-1} \tag{85}$$

It should be noted that the experimental temperature, for this system, proves to be negative, and that the same is true (as is required) of the pressure.

With T and p available, it becomes possible to compare P_v, given by Eq. (80), with the experimental (solid) curve in Fig. 3. The dashed curve in the figure is P_v from Eq. (80), calculated using Eqs. (81), (84), and (85). Although the agreement is not perfect, the two curves are similar enough to lend credibility to the theory. One can use one's eye to easily invert both the positive and negative noisy fluctuations in the solid (experimental) curve, so that the two curves are brought nearly into coincidence. If the data are only incomplete, in the sense that not a large enough number of cars has been included in the sample, then studies involving a larger number might bring the curves directly into coincidence. On the other hand, if the system is nonergodic, or if other unrecognized factors are operative, then no amount of augmentation of the sample will accomplish this.

In closing this section we note that the values of T and p are obtained, from the values of $\bar{l}$ and $\bar{v}$, by using Eqs. (82) and (83). These equations assure that the relations between the parameters are "thermodynamically valid." Therefore, the same relation should be implicit in the equation of state curves drawn in Fig. 1. It is satisfying to note that this is the case. If the value of $\bar{l}$ obtained from Table II of Edie *et al.* is used with the value of T in Eq. (84), in connection with the curves of Fig. 1, it is discovered that the appropriate isotherm gives the value of p appearing in Eq. (85). Thus the internal consistency of the procedure is demonstrated.

The distribution, Eq. (80), using the Holland Tunnel parameters, very closely resembles a Gaussian. In fact when this resemblence is quantified by expanding $\ln P_v$ in terms of $v-\bar{v}$, keeping only quadratic terms, the resulting Gaussian is almost indistinguishable from the dashed curve in Fig. 3. Figure 4 exhibits the comparison.

In the Gaussian approximation it is easy to show that the square root of the fluctuation is given as follows:

$$\langle (v-\bar{v})^2 \rangle^{1/2} = (|T| \lambda_0)^{1/2} \tag{86}$$

so that the "width" of the distribution is given by the geometric mean of the collective and individual sensitivities.

12. PAIR CORRELATION FUNCTION

The use of Eqs. (29) and (56) coupled to Eq. (52) permits us to learn something about the "structure" of the traffic. Some study of this structure

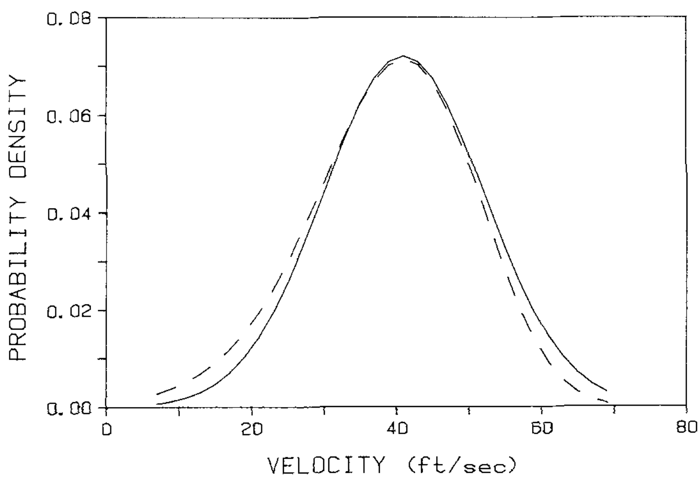

Fig. 4. Gaussian and theoretical velocity distributions. Solid curve: Gaussian approximation to P_v of Eq. (80), where $\bar{v} = 41.00$ ft sec^{-1} and $T = -4.458$ ft sec^{-1}. Dashed curve: normalized probability density, P_v of Eq. (80), where $\lambda_0 = 27.79$ ft sec^{-1}, $a = 30.34$ ft, $p = -0.2090$ sec^{-1}, and $T = -4.441$ ft sec^{-1}.

was attempted in Edie *et al.* (see also Ref. 17a), and only tentative results were obtained. In any event, we are not able to interpret those results using the present method. We can, however, investigate several different measures of the structure, including (1) the distribution of various nearest neighbors (first nearest neighbor, second neighbor, etc.) following a particular car, and (2) the probability that *any* following car is located at a given distance from a particular car. The second measure is, in fact, the standard pair correlation function.

If we pass to the continuum, in Eqs. (29) and (56), and let u become infinitesimal while, at the same time, using Eq. (52) to express nu in terms of $\bar{l}$, we easily obtain the space distribution function of the first nearest following car. If we also replace α and β in Eq. (29) by making use of Eqs. (30) and (31) we obtain, for the distribution of this first nearest neighbor, the result

$$P_1(l)\,dl = Kl^{-(\lambda_0/T)-1}e^{-pl/T}\,dl \tag{87}$$

where K is a normalization constant, and the quantity on the left is the probability that the nearest "following" neighbor lies at l in the range dl.

The distribution of the second nearest neighbor with respect to the first nearest neighbor is given by an expression, fully identical with that in Eq. (87), except that now l refers to the distance between the first and the second nearest neighbors. The probability of position x, of the center of the second nearest neighbor, with respect to the center of the original car, is then obtained as the product of the probabilities of location of the first and second nearest neighbors summed over all locations of the first neighbor, the center of the second nearest neighbor being held fixed at x. Thus we obtain

$$P_2(x)\,dx = \left[\int_a^{x-a} P_1(x-\xi)\,P_1(\xi)\,d\xi\right]dx \tag{88}$$

where $P_2(x)\,dx$ is the probability that the second nearest neighbor lies in the range dx, at a distance x behind the original car. Continuing in this manner we can show, in general, that

$$P_j(x)\,dx = \left[\int_{(j-1)a}^{x-a} P_1(x-\xi)\,P_{j-1}(\xi)\,d\xi\right]dx \tag{89}$$

The limits in the integrals of Eqs. (88) and (89) are determined by the fact that the $(j-1)$st car can have its center no closer to the center of the jth car than a distance a, and that its center must be at least a distance $(j-1)a$ behind the center of the original car.

By repeated application of the convolution theorem for the Laplace transform to the relation, Eq. (89), we find that

$$P_j(t+ja) = \mathscr{L}^{-1}(\{\mathscr{L}[P_1(t+a)]\}^j) \tag{90}$$

in which $\mathscr{L}$ signifies the Laplace transform while $\mathscr{L}^{-1}$ represents its inverse, and where

$$\mathscr{L}[P_1(t+a)] = \int_0^\infty e^{-st} P_1(t+a)\, dt \tag{91}$$

In Eq. (87) it is convenient to replace $-(\lambda_0/T)-1$ by h and p/T by α. Then K is determined by the relation

$$K = \left(\int_a^\infty l^h e^{-\alpha l}\, dl\right)^{-1} \tag{92}$$

In the special case that h is an integral, the integral in this equation can be evaluated leading to the result

$$K = \left[e^{-\alpha a} \sum_{k=0}^{h} (h!/k!)(a^k/\alpha^{h-k+1})\right]^{-1} \tag{93}$$

In this case it is also possible to evaluate the Laplace transform appearing in Eq. (91) and we obtain

$$\mathscr{L}[P_1(t+a)] = \left\{\sum_{k=0}^{h} (h!/k!)[a^k/(s+\alpha)^{h-k+1}]\right\} \Big/ \left[\sum_{k=0}^{h} (h!/k!)(a^k/\alpha^{h-k+1})\right] \tag{94}$$

This equation may be substituted into Eq. (90) leading to an extensive sum of terms. However, each of these terms may be inverted (in the Laplace transform sense) so that an analytical expression for $P_j(t+ja)$ is obtained.

If we are interested in deriving the pair correlation function which gives the density of cars in dx, a distance x behind the original car, that density is composed of the sum of the densities of all successive neighbors, since we need not specify which neighbor is involved. Thus we find for the pair correlation function,

$$P(x) = \sum_{j=1}^{\infty} P_j(x) \tag{95}$$

However, the sum on the right will not contain an infinite number of terms, unless $j=\infty$. Any particular distance x can only accommodate a finite number of cars, i.e., equal to the number of times that the halting distance a, can be fit into x. Thus P_j for j larger than this number must vanish. In any event the method for determining $P(x)$ involves, first evaluating the various P_j and substituting them into Eq. (95).

In the special case $h=0$, Eqs. (93) and (94) become, respectively,

$$K = \alpha e^{\alpha a} \tag{96}$$

$$\mathscr{L}[P_1(t+a)] = \alpha/(s+\alpha) \tag{97}$$

where s is the transform parameter. Substituting Eq. (97) into Eq. (90) leads, for the case $h=0$, to the result

$$P_j(x) = \alpha^j[(x-ja)^{j-1}/(j-a)!]\,e^{-\alpha(x-ja)} \tag{98}$$

This equation can be substituted into Eq. (95) to arrive at the pair correlation function for $h=0$.

In Fig. 5 we plot $P(x)$ for a case in which h is equal to zero, α is equal to 0.047055, and a, equal to 30.34 ft. We recall that $\alpha = p/T$. In the figure, $P(x)$ is plotted vs. x/a. It can be seen that the pair correlation function exhibits damped oscillations, implying short-range order, just as in the case of the pair correlation function for molecules in a fluid. We also note that, for this particular case in which $h=0$, the correlation function starts with a

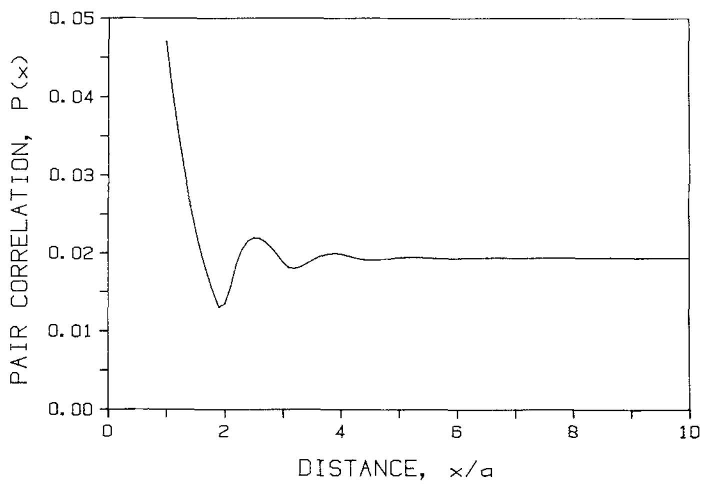

Fig. 5. Pair correlation function simulating equality of "collective" and "individual" sensitivities. Pair correlation function, $P(x)$ of Eq. (95), versus reduced distance ($a=30.34$ ft) for the case $h=0$. Calculations employed Eq. (98) with $\alpha=0.047055$. This situation corresponds to $T=-\lambda_0$, with $\lambda_0=27.79$ ft sec^{-1}, predicting $\bar{v}=12.58$ ft sec^{-1} and $\bar{l}=51.05$ ft.

negative slope at the halting distance, a. The prediction of the correlation function, for this case, is that the most probable position of the nearest neighbor is at the halting distance. This runs counter to one's intuition, since it implies that the following car tends, excessively, to "tailgate" the leading car. However, a value of $h=0$ corresponds to an unrealistic situation, namely, one in which the exponent of l in Eq. (87) vanishes. This only occurs when

$$T = -\lambda_0 \tag{99}$$

Since $|T|$ is the "collective" sensitivity, while λ_0 is the "individual" sensitivity, Eq. (99) implies that the two sensitivities are equal! However, it is only reasonable to assume that the "collective" sensitivity, since it involves the cooperation of a large number of drivers, will be smaller than the "individual" one. In that case (when T is negative), h will be positive. The maximum in P_1 occurs at

$$l = l_{\max} = hT/p \tag{100}$$

and since the ratio T/p is positive, this means that the maximum occurs at increasingly larger positive values of l as $-T$ decreases and h increases. One expects, because of this, that the maximum will exceed a, so that the most probable position for the nearest neighbor will be located at distances larger than the halting distance. As we shall see, this is true for the data in Table II of Edie *et al.*

We now turn to this data, and attempt to calculate the pair correlation function for the line of cars studied in the Holland Tunel. For this case, T and p are given by Eqs. (84) and (85). Furthermore, λ_0 is 27.79 ft sec^{-1}. From these data one readily finds

$$h = 5.258 \approx 5 \tag{101}$$

and

$$\alpha = 0.047055 \tag{102}$$

If h were approximated by the integer 5, we could use Eqs. (93) and (94) together with Eq. (90) to determine P_j. It turns out that the inversion specified by Eq. (90) can actually be performed analytically. However, the resulting expression is such a complicated sum, consisting of so many terms, that it is simpler to return to Eq. (88), and to solve the problem iteratively by first computing P_2 from P_1, P_3 from P_2, etc. Of course, in this process, the integral in Eq. (89) is evaluated numerically and the actual value for h is used.

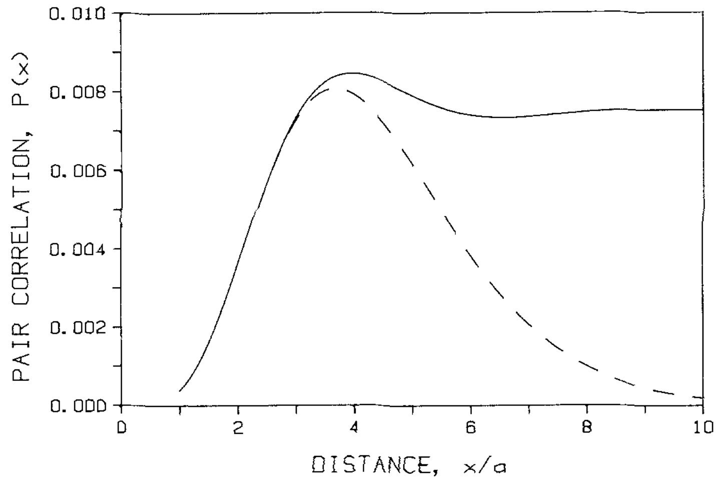

Fig. 6. Pair correlation function for the Holland Tunnel traffic data of Edie *et al.* Solid curve: pair correlation function, $P(x)$ of Eq. (95), versus reduced distance ($a = 30.34$ ft). Calculations employed Eqs. (87) and (89) with the values of $p = -0.2090\ \mathrm{sec}^{-1}$ and $T = -4.441\ \mathrm{ft\ sec}^{-1}$ determined from the data of Edie *et al.* This situation corresponds to $T = -0.16\ \lambda_0$, with $\lambda_0 = 27.79\ \mathrm{ft\ sec}^{-1}$, predicting $\bar{v} = 38.88\ \mathrm{ft\ sec}^{-1}$ and $\bar{l} = 133.2$ ft (experimental data: $\bar{v} = 38.88\ \mathrm{ft\ sec}^{-1}$ and $\bar{l} = 133.2$ ft). Dashed curve: nearest-neighbor distribution, $P_1(x)$ of Eq. (87), for the Holland Tunnel data.

A plot of $P(x)$, for this case, obtained in this way, is show in Fig. 6. Again, the abscissa is x/a rather than x itself. As indicated in Eq. (81), a is 30.34 ft.

The dashed curve in Fig. 6 is a plot of P_1, the distribution of the nearest neighbor for this case. $P(x)$ is, of course, given by the solid curve. One notes that the amplitude of oscillation is much smaller than that for the case illustrated by Fig. 5. In fact, the second oscillation, which occurs in the neighborhood of $x/a = 9$, is hardly discernible in the plot. However, the actual numbers show it to be there. More important is the fact that the first maximum, unlike the case of Fig. 5, is at $x/a = 4$. Thus the distribution *does not* start, at the halting distance, with a negative slope. This indicates, as anticipated earlier, that the "collective" is much smaller than the "individual" sensitivity. Until the first maxmimum is reached, at $x/a = 4$ the dashed curve for P_1 shows the distribution to be almost entirely accounted for by the *first* nearest neighbor. Beyond this point the second and further neighbors become dominant. We also note that the first maximum occurs at about the average spacing between cars which, for this case, is $x/a = 4.39$. Also the curve levels off, at large distances, at the observed average density of $0.00750\ \mathrm{ft}^{-1}$.

The pair correlation function appearing in Fig. 6 is consistent with the typical driver's perception (at typical velocities) of the traffic behind him. The main feature of the structure involves the car directly behind, and, beyond that, cars rapidly become uncorrelated. The damping of the oscillations appears to be connected with the disparity between the "individual" and "collective" sensitivities. The smaller the latter, the more chaotic, and less structured, is the traffic.

We can still speculate on whether T can achieve *positive* values. The integral in Δ_0, Eq. (60), will converge when T is positive, as along as p is *also* positive. Now, λ_0 and a are parameters which refer to the behavior (on a particular road) of an *individual* driver. In contrast, T and p depend on the *collective* behavior of the drivers, and are not derivable from theory alone. They indeed reflect aspects of behavior which are not contained within λ_0 and a. As a result they must be *measured*, in the same way that we obtained the values for the Holland Tunnel, i.e., from an analysis of the data.

The question of *positive* temperature was mentioned briefly in Section 7. Unfortunately, apparently no data exist for the situations in which T and p are positive. We did however guess at the features which might give rise to such situations. A positive T implies that, at constant $\bar{l}$, the disorder *increases* as $\bar{v}$, the average velocity, increases. As indicated earlier, under normal driving circumstances we would expect the drivers to *order* themselves, in the interest of safety, as they increase their average velocity. Thus we expect T to be negative, and this seems to be true of the Holland Tunnel data.

But as discussed in Section 7, there is one situation in which we might expect the opposite result, namely, a traffic jam in which all the cars are moving very slowly, and are almost bumper to bumper. In the limit of no movement, the cars would from a "lattice" of lattice parameter, a. They would thus form a one-dimensional, ordered "crystal" in which the entropy would be zero. As they begin to move, a certain amount of disorder *must* develop, and there must occur an increase of entropy. Thus we would have a case in which the entropy would indeed *increase* with an *increase* of average velocity. This suggests that we look for positive T in congested traffic situations in which the average velocity is *small* and the average density is *high*.[7]

With this is mind we choose a case in which $\bar{v} = 2.967 \text{ ft sec}^{-1}$ and $\bar{l} = 34.02$ ft. Thus $\bar{v}$ is small compared to $\lambda_0 = 27.79 \text{ ft sec}^{-1}$, the value for the Holland Tunnel, and $\bar{l}$ is only slightly larger than the halting distance,

[7] A higher temperature has been hypothesized to correspond to greater traffic congestion by Wilson (Ref. 28).

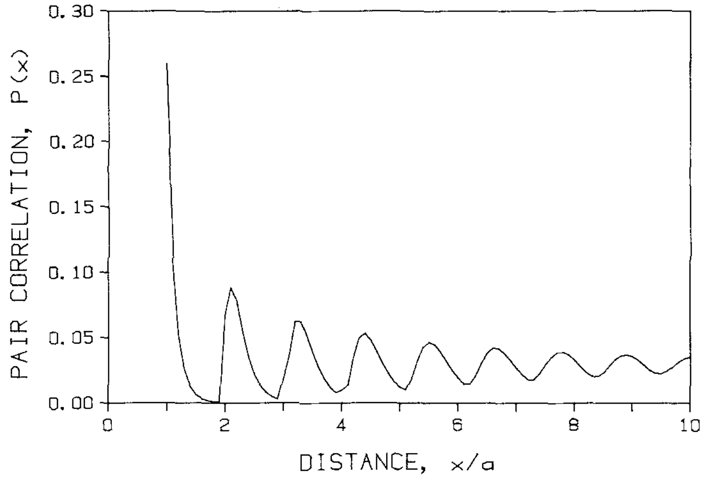

Fig. 7. Pair correlation function simulating congested traffic. Pair correlation function, $P(x)$ of Eq. (95) versus reduced distance ($a = 30.34$ ft). Calculations employed Eqs. (87) and (89) with $p = 0.2090\ \text{sec}^{-1}$ and $T = 4.441$ ft sec^{-1}. This situation corresponds to $T = 0.16\,\lambda_0$, with $\lambda_0 = 27.79$ ft sec^{-1} predicting $\bar{v} = 2.967$ ft sec^{-1} and $\bar{l} = 34.02$ ft.

$a = 30.34$ ft, for the Holland Tunnel. These values of $\bar{v}$ and $\bar{l}$ together with the Holland Tunnel values of λ_0 and a require $T = 4.441$ ft sec^{-1} and $p = 0.2090\ \text{sec}^{-1}$, both *positive.* For these values of p and T we can calculate the pair correlation function, just as we did for Fig. 6, for the measured values of p and T. The result is shown in Fig. 7. We see that there are indeed many oscillations, and that order extends to a much longer range than in the case of Fig. 6. The picture does give the impression of a crystal in the process of "melting."

Since no experimental data exist for single-lane traffic at velocities and densities similar to those on which Fig. 7 is based, the discussion surrounding it, although interesting, remains speculative.

13. SUMMARY

The main points of this paper are the following:

(1) Although entropy should be definable in nonphysical systems it should not be chosen arbitrarily as a measure of uncertainty. Rather it should arise naturally in a development which relates it, simply, to other measurables.

(2) The most natural procedure for accomplishing this involves the maximum entropy formalism.

(3) The maximum entropy formalism has been shown, by Levine and his coworkers and by Shore and Johnson, to be the unique algorithm which together with known information (constraints) allows the *consistent* inference of probability distributions of independent events. Furthermore the distribution can be inferred, using consistency, without ever defining entropy or maximizing anything! As a result, the maximum entropy formalism becomes merely a convenient procedure for generating the consistent distribution. However, as an assumption, it achieves a distinction which other assumptions do not have, *because it is* consistent.

(4) Entropy though useful need not, and should not, occupy a primary position in a "thermodynamics" of a nonphysical system. Rather the methodology of the "thermodynamics," and especially the *undetermined multipliers* which may arise during its application, will have primary value.

(5) We develop a thermodynamics, and configure it so that it is useful for the treatment of a system of vehicular traffic confined to a single lane. In particular, the interaction between cars is chosen to be one of those of the "follow-the-leader" variety, which has been suggested in the literature, and subjected to some experimental verification. However, it is emphasized that the thermodynamic framework is independent of the mode of interaction.

(6) The equation of state for the traffic is derived. For negative temperature (collective sensitivity), it is shown that the Helmholtz free energy is *reduced* by fluctuations in local density. This, as well as negative temperature, is consistent with intuition. It is also shown that the "passage to the continuum" is essentially achieved when the quantum of velocity is less than the "individual" sensitivity.

(7) Experimental data from the Holland Tunnel are analyzed in terms of our development and traffic temperature (collective sensitivity) and traffic pressure are determined. The temperature is negative, as expected on the basis of intuition.

(8) The theory allows one to evaluate the nearest-neighbor and pair correlation functions for the cars. Such correlation functions are actually evaluated for the Holland Tunnel data.

(9) A hypothetical situation in which both average velocity and average headway are small is investigated. For this case the traffic temperature proves to be positive (again consistent with intuition) and the pair correlation function resembles that of a melting crystal, i.e., a traffic jam which is beginning to dissolve (again consistent with intuition).

(10) It is emphasized that the thermodynamics and the ther-

modynamic parameters are *analogs*. The parameters (usually undetermined multipliers) have to be *interpreted* and *measured*. However, they prove to be useful summarizers of the collective behavior of the traffic.

14. NOTE IN CLOSING

One of the authors of this paper, Professor E. W. Montroll, sadly passed away during the course of this work. Although in truth, Professor Montroll was not given the time to contribute heavily to the development of the paper, it would never have been started without his energy and inspiration.

Another of the authors (H. Reiss) visited the Institute for Physical Science and Technology at the University of Maryland, in the fall of 1982, in order to collaborate with Montroll in research on the kinetics of the two-dimensional ferromagnet. Professor Montroll was "the last of the natural philosophers," and when Reiss arrived, Montroll was pondering, among many other things, one of his recent papers, titled "On the Entropy Function in Sociotechnical Systems." In this paper, he was looking for entropy (of some sort) in both the single-lane traffic system, and in, of all things, the Sears Roebuck catalog. He knew it was there, but he had not quite found it in a form with which he was comfortable. Montroll's curiosity was irresistible, and the visitor found himself working on *traffic* and not on the *two-dimensional ferromagnet*. Not much was lost however, and much was gained, because this led to the inception of the present paper.

In addition to his many great contributions to mathematical physics, Professor Montroll had, for years, been interested in the application of physical science methodology to the solution of social science problems. His book, with Badger, *Introduction to Quantitative Aspects of Social Phenomena*, is an example of this. Among the things which interested him, in social and economic systems, was the possible role of entropy. Thus this paper goes beyond having him as a coauthor; it is dedicated to his memory.

ACKNOWLEDGMENT

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